

# BIVARIANT HOPF CYCLIC COHOMOLOGY

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**ABSTRACT.** For module algebras and module coalgebras over an arbitrary bialgebra, we define two types of bivariant cyclic cohomology groups called bivariant Hopf cyclic cohomology and bivariant equivariant cyclic cohomology. These groups are defined through an extension of Connes' cyclic category  $\Lambda$ . We show that, in the case of module coalgebras, bivariant Hopf cyclic cohomology specializes to Hopf cyclic cohomology of Connes and Moscovici and its dual version by fixing either one of the variables as the ground field. We also prove an appropriate version of Morita invariance for both of these theories.

## 1. INTRODUCTION

In this paper we define two types of bivariant cyclic cohomology groups  $HC_{\text{Hopf}}^*(A, A'; M, M')$ , and  $HC_H^*(A, A'; M, M')$  where  $H$  is a bialgebra,  $A$  and  $A'$  are  $H$ -module algebras and  $M$  and  $M'$  are stable  $H$ -module/comodules. The modules  $M$  and  $M'$  appear as coefficients to twist the cyclic  $H$ -modules of  $A$  and  $A'$ , respectively. We refer to the first theory as bivariant Hopf cyclic cohomology and to the second theory as bivariant equivariant cyclic cohomology theory. We also define bivariant groups  $HC_{\text{Hopf}}^*(C, C'; M, M')$  and  $HC_H^*(C, C'; M, M')$ , where  $C$  and  $C'$  are  $H$ -module coalgebras. These latter groups have much to do with the Hopf cyclic cohomology of Hopf algebras first defined by Connes and Moscovici [4, 5, 6] and its extensions developed in [8, 9, 12, 11]. More precisely, we show that for  $C = k$  the ground field and  $M = k$  the trivial module, we have canonical isomorphisms between  $HC_{\text{Hopf}}^*(k, C'; k, M')$  and the Hopf cyclic cohomology of the  $H$ -module coalgebra  $C'$  with coefficients in  $M'$  as defined in [11]. It is shown in [11] that when  $H$  is a Hopf algebra and  $M$  is an stable anti-Yetter-Drinfeld module this group reduces to the Hopf cyclic cohomology of  $C$  defined in [8]. In particular for  $C' = H$  a Hopf algebra, our theory specializes to Connes-Moscovici's Hopf cyclic cohomology for Hopf algebras. If, on the other hand, we specialize the second leg to  $C' = k$  and  $M' = k$ , the groups  $HC_{\text{Hopf}}^*(C, k; M, k)$  are the dual Hopf cyclic cohomology of  $C$  with coefficients in  $M$ . Here by duality we mean Connes' duality for the cyclic category  $\Lambda$  [1, 3] whose relevance to Hopf cyclic cohomology was first noticed in [13].

To define our bivariant groups we use the Ext bifunctor and Connes' cyclic category  $\Lambda$  in an appropriate  $H$ -equivariant setting. The bivariant equivariant theory is a non-trivial extension of the Hopf cyclic (co)homology in that there are spectral sequences (Propositions 5.8 and 5.9) converging to it where Hopf cyclic cohomology appears in the  $E_2$ -term.

Our definitions immediately imply that for both theories we have associative composition, or Yoneda products. There is also an analogue of Connes' periodicity  $S$ -operator in our context coming from an

external product: there are graded actions of the algebras  $HC_H^*(k, k; k, k)$  and  $HC_{\text{Hopf}}^*(k, k; k, k)$  on every bivariant equivariant and bivariant Hopf cyclic cohomology modules

$$\begin{aligned} HC_H^p(k, k; k, k) \otimes HC_H^q(X, X', M, M') &\rightarrow HC_H^{p+q}(X, X'; M, M') \\ HC_{\text{Hopf}}^p(k, k; k, k) \otimes HC_{\text{Hopf}}^q(X, X', M, M') &\rightarrow HC_{\text{Hopf}}^{p+q}(X, X', M, M') \end{aligned}$$

where the algebra  $HC_{\text{Hopf}}^*(k, k; k, k)$  is the  $k$ -algebra generated by the  $S$ -operator. On the other hand, the algebra  $HC_H^*(k, k; k, k)$  is a combination of  $\text{Ext}_H^*(k, k)$  and the  $k$ -algebra generated by the  $S$ -operator via a spectral sequence of graded  $k$ -algebras (Corollary 3.12). Thus, one can see that bivariant equivariant cyclic cohomology is a natural extension of both cyclic (co)homology and ordinary (co)homology of  $H$ -modules in a unified theory.

The original motivation to develop a bivariant cyclic cohomology [10, 15, 7], first defined by Connes in [1, 2] as an Ext group, was to define a receptacle for a bivariant Chern-Connes character defined on (smooth) cycles of KK-theory. There is a similar question in our Hopf setting as well but we will not address it in this paper.

Here is a plan of this paper. In Section 2 we define the category of (co)cyclic  $H$ -modules and give a reinterpretation of these modules, and their variations, as left and right modules over a short list of algebras all of which can be defined in terms of a single large algebra  $\mathcal{P}(H)$ . In Section 3 we prove some results in homological algebra for the algebra  $\mathcal{P}(H)$  and its modules we defined in Section 2. In Section 4 we define a cyclic  $H$ -module associated with an  $H$ -module algebra with coefficients in a stable  $H$ -module/comodule. Using this object we define bivariant Hopf and bivariant equivariant cyclic homology of a pair of  $H$ -module algebras with coefficients in an arbitrary pair of stable  $H$ -module/comodules. By using the results of Section 3, we also show how our theories relate to the ordinary cyclic homology and cohomology of algebras, in combination with the cohomology of these algebras viewed simply as  $H$ -modules. In Section 5 we develop the same theory for module coalgebras and we also investigate the connections of these bivariant theories with Hopf cyclic cohomology. In Section 6 we develop the notion of  $H$ -categories and a cyclic homology theory for  $H$ -categories. We show that (co)cyclic  $H$ -module associated with a module (co)algebra can also be interpreted as an  $H$ -categorical invariant, which leads us to the Morita invariance.

Throughout this paper we assume  $k$  is a field and  $H$  is an associative/coassociative, unital/counital bialgebra, or a Hopf algebra with an invertible antipode whenever it is necessary. By a (say, left-left)  $H$ -module/comodule we mean a left  $H$ -module which is also a left  $H$ -comodule with no compatibility assumption between action and coaction.

## 2. THE CATEGORY OF CYCLIC $H$ -MODULES AND ITS VARIANTS

In this section we define the categories of cyclic and cocyclic  $H$ -modules where  $H$  is a bialgebra. We also define closely related categories of para-(co)cyclic and pseudo-para-(co)cyclic  $H$ -modules. We show that when  $H$  is a Hopf algebra, Connes' fundamental isomorphism between the cyclic category and its dual [1],

can be extended to all of the above categories. This plays an important role in our definition of bivariant Hopf cyclic groups.

We denote Connes's cyclic category by  $\Lambda$ . Recall from [1, 3, 14] that a cyclic (resp. cocyclic) object in a category  $\mathcal{C}$  is a contravariant (resp. covariant) functor  $X_\bullet : \Lambda \rightarrow \mathcal{C}$ . A cyclic (resp. cocyclic)  $k$ -module is simply a cyclic (resp. cocyclic) object in the category of  $k$ -modules. Given a cyclic module  $X_\bullet$  we denote its cyclic homology groups by  $HC_*(X_\bullet)$ . Similarly we write  $HC^*(X_\bullet)$  to denote the cyclic cohomology of a cocyclic module  $X_\bullet$ .

A remarkable property of Connes' cyclic category is its self duality. It is shown in [1] that there is a natural isomorphism of categories  $\Lambda \rightarrow \Lambda^{op}$ . This fact plays almost no role in the cyclic homology of algebras or coalgebras, but it is of considerable importance in Hopf cyclic cohomology as it was first observed in [13]. In the sequel the cyclic duality unavoidably manifests itself in the Specialization Theorem (Theorem 5.5).

**Definition 2.1.** Let  $H$  be a bialgebra. A (co)cyclic module  $X_\bullet$  is called a (co)cyclic  $H$ -module if (i) for each  $n \geq 0$ , the module  $X_n$  is a (left) right  $H$ -module and (ii) all the (co)cyclic structure morphisms commute with the action of  $H$ . A para-(co)cyclic  $H$ -module is the same as a (co)cyclic  $H$ -module except that the cyclic operators  $\tau_n$  do not necessarily satisfy  $\tau_n^{n+1} = id_n$  for  $n \geq 0$ . We may even drop the condition that  $\tau_n$  is invertible, if necessary.

To define our bivariant groups we have to reinterpret the (co)cyclic  $H$ -modules and its relatives as modules over certain (non-unital and non-counital) bialgebras defined below. Here we define them as algebras but we are going to prove in Proposition 3.10 that they really are non-unital and non-counital bialgebras.

**Definition 2.2.** Let  $H[\Lambda_{\mathbb{N}}]$  be the algebra generated by  $H$ -linear combinations of the symbols  $\partial_i^n$ ,  $\sigma_j^m$  and  $\tau_n^\ell$  for  $0 \leq n$ ,  $0 \leq m$ ,  $0 \leq i \leq n+1$ ,  $0 \leq j \leq m$  and  $\ell \in \mathbb{N}$  satisfying

$$(2.1) \quad \partial_i^{n+1} \partial_j^n = \partial_{j+1}^{n+1} \partial_i^n \quad \text{and} \quad \sigma_j^{n-1} \sigma_i^n = \sigma_i^{n-1} \sigma_{j+1}^n \quad \text{for } i \leq j \quad \text{and} \quad \tau_n^s \tau_n^t = \tau_n^{s+t} \quad \text{for } s, t \in \mathbb{N}$$

$$(2.2) \quad \sigma_i^n \partial_i^n = \sigma_i^n \partial_{i+1}^n = \tau_n^0 \quad \text{and} \quad \partial_i^n \sigma_j^n = \begin{cases} \sigma_{j+1}^{n+1} \partial_i^{n+1} & \text{if } i \leq j \\ \sigma_j^{n+1} \partial_{i+1}^{n+1} & \text{if } i > j \end{cases}$$

$$(2.3) \quad \partial_i^n \tau_n^j = \tau_{n+1}^{i+p} \partial_q^n \quad \text{where } (i+j) = (n+1)p + q \quad \text{with } 0 \leq q \leq n$$

$$(2.4) \quad \tau_n^i \sigma_j^n = \sigma_q^n \tau_{n+1}^{i+p} \quad \text{where } (-i+j) = (n+1)(-p) + q \quad \text{with } 0 \leq q \leq n$$

All other products between the generators are 0 and the product is extended  $H$ -linearly. We also define another algebra  $H[\Lambda_{\mathbb{Z}}]$  where we allow  $\tau_n^\ell$  with  $\ell \in \mathbb{Z}$ . Define  $H[\mathcal{D}]$  as the subalgebra generated by  $\partial_j^m$  and  $H[\mathcal{D}_{\mathbb{N}}]$  as the subalgebra generated by  $\tau_n^i$  and  $\partial_j^m$  for all possible  $n, m, j$  and  $i \geq 0$ . We finally define  $H[\Lambda_+]$  as the subalgebra generated by  $\sigma_j^n$  and  $\partial_j^n$  where we only require  $0 \leq j \leq n$  for any  $n \geq 0$ .

**Definition 2.3.** Define a  $k$ -algebra  $\mathcal{P}(H)$  as an amalgamated product of  $k$ -algebras

$$\mathcal{P}(H) := k[\Lambda_{\mathbb{Z}}] \underset{k[\Lambda_+]}{*} H[\Lambda_+]$$

It is the  $k$ -algebra generated by the same generators and relations as  $H[\Lambda_{\mathbb{N}}]$  however we drop the relations  $[x, \tau_n^i] = 0$  for any  $x \in H$  and for all possible  $n \geq 0$  and  $i$ . Any left  $\mathcal{P}(H)$ -module is called a pseudo-para-cocyclic  $H$ -module and any right  $\mathcal{P}(H)$ -module is called a pseudo-para-cyclic  $H$ -module. We define another algebra  $\mathcal{P}(H)'$  by  $k[\Lambda_{\mathbb{N}}] \underset{k[\Lambda_+]}{*} H[\Lambda_+]$  where we drop the condition that  $\tau_n$  is invertible for  $n \geq 0$ .

**Remark 2.4.** The difference between  $H[\Lambda_{\mathbb{Z}}]$  and  $\mathcal{P}(H)$  is that for para-(co)cyclic modules (i.e. modules of  $H[\Lambda_{\mathbb{Z}}]$ ), the action of elements of  $H$  and the action of the generators  $\tau_n^\ell$  do necessarily commute while for pseudo-para-(co)cyclic (i.e. modules of  $\mathcal{P}(H)$ ) this is not required.

**Definition 2.5.** Let  $H[\Lambda]$  be the quotient of  $H[\Lambda_{\mathbb{N}}]$  by the right (which is also a bilateral) ideal generated by elements of the form  $(\tau_n^{n+1} - \tau_n^0)$  for all  $x \in H$  and for all possible  $n$  and  $i$ . Define also  $H[\mathcal{D}_C]$  as the image of  $H[\mathcal{D}_{\mathbb{N}}]$  under this quotient.

**Definition 2.6.** Let  $\mathcal{A}$  be one of  $\mathcal{P}(H)$ ,  $\mathcal{P}(H)'$ ,  $H[\Lambda_{\mathbb{N}}]$ ,  $H[\Lambda_{\mathbb{Z}}]$ ,  $H[\Lambda]$ ,  $H[\mathcal{D}_{\mathbb{N}}]$  and  $H[\mathcal{D}_C]$ . A left (or right)  $\mathcal{A}$ -module  $X_\bullet$  is called faithful if for every  $x \in X_\bullet$  there exists a finite number of elements  $y_i \in X_\bullet$  and integers  $n_i \in \mathbb{N}$  such that  $x = \sum_i \tau_{n_i}^0 y_i$  (resp.  $x = \sum_i y_i \tau_{n_i}^0$ ).

Throughout the paper, we will assume that all of our modules over the algebras  $\mathcal{A}$  considered above are faithful.

**Lemma 2.7.** *Let  $\mathcal{A}$  be one of the algebras  $H[\Lambda_{\mathbb{Z}}]$ ,  $H[\Lambda]$ ,  $H[\mathcal{D}_{\mathbb{N}}]$ ,  $H[\mathcal{D}_C]$ . Then the category of faithful right (left)  $\mathcal{A}$ -modules is isomorphic to the category of para-(co)cyclic, (co)cyclic, pre-para-(co)cyclic and pre-(co)cyclic  $H$ -modules, respectively.*

*Proof.* We will give the proof for pre-para-cocyclic case but the proof for the other cases are very similar. Assume  $X_\bullet$  is a pre-para-cocyclic  $H$ -module. Then  $X_\bullet$  is a  $\mathbb{N}$ -graded  $H$ -module with structure morphisms  $\partial_j^n$ , and  $\tau_n^\ell$  where  $\ell, n \in \mathbb{N}$  and  $0 \leq j \leq n+1$  which satisfy the conditions stated in Equations (2.1) through (2.4) and the condition that  $\tau_n^0(x) = x$  for any  $x \in X_n$ . In other words  $X_\bullet$  is a faithful left  $H[\mathcal{D}_{\mathbb{N}}]$ -module. Conversely, assume  $X_\bullet$  is a faithful left  $H[\mathcal{D}_{\mathbb{N}}]$ -module. Define  $X_n$  as the submodule of  $X_\bullet$  consisting of elements  $x$  such that  $x = \tau_n^0(y)$  for some  $y \in X_\bullet$ . Note that since  $\tau_n^0$  is idempotent  $\tau_n^0(x) = x$  for any  $x \in X_n$ . Moreover, since  $X_\bullet$  is faithful  $X_\bullet = \bigoplus_n X_n$  by definition. The actions of the generators  $\partial_j^n$  define a pre-cosimplicial structure on  $X_\bullet$ . For every  $n \geq 0$  we also have an action of  $\mathbb{N}$  on  $X_n$  via  $\tau_n^\ell$ . We must check that combination of these structure morphisms do really define a pre-para-cocyclic structure. Consider Equation (2.3) for  $i = n+1$  and  $j = 0$  and we see that since  $i+j = n+1 = (n+1)1+0$  we must have

$$\partial_{n+1}^n = \tau_{n+1} \partial_0^n$$

Moreover, we observe that since  $j + 1 = (n + 1)0 + j + 1$  for  $0 \leq j \leq n - 1$  we get

$$\partial_j^n \tau_n = \begin{cases} \tau_{n+1} \partial_{i+1}^n & \text{if } 0 \leq j \leq n - 1 \\ \tau_{n+1}^2 \partial_0^n & \text{if } j = n \end{cases} = \tau_{n+1} \partial_{j+1}^n$$

i.e.  $X_\bullet$  is a pre-para-cyclic  $H$ -module.  $\square$

From now on we will use the terms left (right)  $\mathcal{P}(H)$ -module,  $H[\Lambda_{\mathbb{Z}}]$ -module,  $H[\Lambda]$ -module,  $H[\mathcal{D}_{\mathbb{N}}]$ -module and  $H[\mathcal{D}_C]$ -modules and pseudo-para-(co)cyclic, para-(co)cyclic, (co)cyclic, pre-para-(co)cyclic and pre-(co)cyclic  $H$ -module interchangeably.

The algebra  $\mathcal{P}(H)$  plays an important role for us since all the algebras we described above can be obtained from  $\mathcal{P}(H)$ . Almost all important properties of  $\mathcal{P}(H)$  will descend on the rest of the algebras.

**Proposition 2.8.** *Let  $\mathcal{A}$  be one of  $\mathcal{P}(H)$ ,  $H[\Lambda_{\mathbb{Z}}]$ ,  $H[\Lambda]$  and  $H[\mathcal{D}_C]$ . If  $H$  is a Hopf algebra with an invertible antipode then  $\mathcal{A}$  is isomorphic to its opposite algebra via an isomorphism  $\mathcal{A} \xrightarrow{\gamma} \mathcal{A}^{op}$ .*

*Proof.* We will give the isomorphism on  $\mathcal{P}(H)$ . The isomorphism for the other cases is obtained from this isomorphism. We define  $\mathcal{P}(H) \xrightarrow{\gamma} \mathcal{P}(H)^{op}$  by defining it on generators as

$$\gamma(\partial_j^n) = \sigma_j^n, \quad \gamma(\sigma_j^n) = \partial_j^n, \quad \gamma(\tau_n) = \tau_n^{-1} \quad \text{and} \quad \gamma(h) = S(h)$$

for  $h \in H$ ,  $0 \leq j \leq n + 1$ . Note that since  $\partial_{n+1}^n = \tau_{n+1} \partial_0^n$ , one does not have a problem of defining  $\sigma_{n+1}^n := \gamma(\partial_{n+1}^n) := \sigma_0^n \tau_{n+1}^{-1}$ . It is routine to check that  $\gamma$  is an isomorphism of algebras, since  $S$  is invertible.  $\square$

**Theorem 2.9.** *Let  $H$  be a Hopf algebra with an invertible antipode. Then the categories of (pseudo-)(pre-)(para-)cocyclic and (pseudo-)(pre-)(para-)cyclic  $H$ -modules are isomorphic.*

*Proof.* We are going to give the proof for pseudo-para-cyclic modules. The proof for the other cases is similar. We define a functor

$$(\cdot)^\vee : \mathcal{P}(H)\text{-}\mathbf{Mod} \rightarrow \mathbf{Mod}\text{-}\mathcal{P}(H)$$

as follows: for every pseudo-para-cocyclic  $H$ -module  $Y_\bullet$ , the module  $Y_\bullet^\vee$  is the same as  $Y_\bullet$ . However, for every  $\Psi \in \mathcal{P}(H)$  and  $y \in Y_\bullet$ , we define  $y\Psi := \gamma(\Psi)y$ . In the opposite direction,  $(\cdot)^\vee : \mathbf{Mod}\text{-}\mathcal{P}(H) \rightarrow \mathcal{P}(H)\text{-}\mathbf{Mod}$  is defined similarly. However, given any para-cyclic  $H$ -module  $X_\bullet$  and  $x \in X_\bullet^\vee$  and  $\Psi \in H[\Lambda_{\mathbb{Z}}]$  we define  $\Psi x := x\gamma^{-1}(\Psi)$ . This way, one can see that  $(Z_\bullet^\vee)^\vee$  is  $Z_\bullet$  itself for any pseudo-para-(co)cyclic module  $Z_\bullet$ .  $\square$

**Corollary 2.10.** *Let  $\mathcal{A}$  be one of  $\mathcal{P}(H)$ ,  $H[\Lambda_{\mathbb{Z}}]$ ,  $H[\Lambda]$ ,  $H[\mathcal{D}_C]$ . Let  $X_\bullet$  and  $Y_\bullet$  be two right  $\mathcal{A}$ -modules. Then one has an isomorphism of  $k$ -modules*

$$\mathrm{Hom}_{\mathcal{A}}(X_\bullet, Y_\bullet) \cong \mathrm{Hom}_{\mathcal{A}}(X_\bullet^\vee, Y_\bullet^\vee)$$

The functor we defined in Theorem 2.9 extends the cyclic duality functor defined by Connes [1]. As is observed in [13] all the classical examples coming from the cyclic and cocyclic modules of algebras and coalgebras are all ‘one-sided’. That is, if  $X$  is a (co)algebra and if  $C_*(X)$  is the ordinary (co)cyclic module associated with  $X$ , then the cyclic dual  $C_*(X)^\vee$  has trivial homology. However, for an arbitrary (co)cyclic  $H$ -module this need not be the case. For example, for a Lie algebra  $\mathfrak{g}$ , both  $\mathcal{C}_\bullet(U(\mathfrak{g}), k)$  (Definition 5.2) and  $\mathcal{C}_\bullet(U(\mathfrak{g}), k)^\vee$  are homologically non-trivial. For example their periodic cyclic (co)homologies are computed in [4] and [12] respectively, and shown to be both isomorphic to the Lie algebra homology of  $\mathfrak{g}$  with trivial coefficients.

### 3. (CO)HOMOLOGY OF (CO)CYCLIC $H$ -MODULES

Our goal in this section is to extend some results of [1] from the category of (co)cyclic  $k$ -modules to the category of (co)cyclic  $H$ -modules.

Let  $\mathcal{A}$  be one of the algebras  $\mathcal{P}(H)$ ,  $H[\mathbb{Z}]$ ,  $H[\Lambda]$ ,  $H[\mathcal{D}_C]$ . We consider the  $\mathbb{N}$ -graded  $k$ -module  $k_\bullet = \bigoplus_{n \geq 0} k$  as a left  $\mathcal{A}$ -module by letting  $H$  act by the counit  $H \xrightarrow{\varepsilon} k$  and by identity for the rest of the generators.

**Proposition 3.1.** *Let  $H = k$ . Then for any cyclic module  $X_\bullet$  we have natural isomorphisms*

$$\mathrm{Ext}_{k[\Lambda]}^*(X_\bullet, k_\bullet^\vee) \cong HC^*(X_\bullet) \qquad \mathrm{Tor}_*^{k[\Lambda]}(X_\bullet, k_\bullet) \cong HC_*(X_\bullet)$$

*Proof.* In [1, Section 4] a projective biresolution of  $k_\bullet^\vee$  ( $k^\#$  in Connes’ notation) is developed. Connes uses this particular resolution to obtain the first part of the Proposition. The second part follows immediately.  $\square$

**Lemma 3.2.** *Assume we have two left  $k[\Lambda]$ -modules, i.e. cocyclic  $k$ -modules,  $X_\bullet$  and  $Y_\bullet$ . Then one has a natural isomorphism of  $k$ -modules of the form*

$$X_\bullet^\vee \otimes_{k[\Lambda]} Y_\bullet \xrightarrow{t} Y_\bullet^\vee \otimes_{k[\Lambda]} X_\bullet$$

*Proof.* The isomorphism is defined by the transposition  $t(x \otimes y) = (y \otimes x)$  for any  $x \otimes y$  in  $X_\bullet^\vee \otimes_{k[\Lambda]} Y_\bullet$ . However, we must prove that the morphism  $t$  is well-defined. For that purpose, we recall that  $X_\bullet^\vee \otimes_{k[\Lambda]} Y_\bullet$  is a quotient of  $X_\bullet^\vee \otimes Y_\bullet$  by the  $k$ -submodule generated by elements of the form  $(x\Psi \otimes y) - (x \otimes \Psi y)$  where  $x \in X_\bullet^\vee$ ,  $y \in Y_\bullet$  and  $\Psi \in k[\Lambda]$ . However, recall from Theorem 2.9 that  $x\Psi := \gamma(\Psi)x$  therefore

$$t((x\Psi \otimes y) - (x \otimes \Psi y)) = (y \otimes \gamma(\Psi)x) - (y\gamma^{-1}(\Psi) \otimes x)$$

However,  $H = k$ ,  $k[\Lambda] \xrightarrow{\gamma} k[\Lambda]^{op}$  is an involution. The result follows.  $\square$

**Remark 3.3.** In the general situation when  $H \neq k$ , Lemma 3.2 would work only if one has (i)  $S^2 = id$  on  $H$  or (ii) one of the modules involved has the property that  $\gamma^2(\Psi)z = \Psi z$  for any  $\Psi$ .  $k_\bullet$  and  $k_\bullet^\vee$  are two important examples of such  $H[\Lambda]$ -modules while  $H_\bullet$  and  $H_\bullet^\vee$  are not unless  $S^2 = id$ .

**Proposition 3.4.** *Let  $H = k$ . Then for any cocyclic module  $Y_\bullet$  we have natural isomorphisms*

$$\mathrm{Tor}_*^{k[\Lambda]}(k_\bullet^\vee, Y_\bullet) \cong HC_*(Y_\bullet^\vee) \quad \mathrm{Ext}_{k[\Lambda]}^*(Y_\bullet, k_\bullet) \cong HC^*(Y_\bullet^\vee)$$

Moreover,

$$\mathrm{Ext}_{k[\Lambda]}^*(k_\bullet, Y_\bullet) \cong HC^*(Y_\bullet)$$

*Proof.* The first assertion easily follows from Lemma 3.2. For the second assertion, we observe that  $(\cdot)^\vee$  is an isomorphism of categories which means one has a natural isomorphism of functors (Corollary 2.10)

$$\mathrm{Hom}_{k[\Lambda]}(\cdot, k_\bullet) \cong \mathrm{Hom}_{k[\Lambda]}((\cdot)^\vee, k_\bullet^\vee) \cong \mathrm{Hom}_k((\cdot)^\vee \otimes_{k[\Lambda]} k_\bullet, k)$$

by using the fact that for the trivial cocyclic module  $k_\bullet$ , the  $k$ -vector space dual and the cyclic dual are isomorphic. Then we use Lemma 3.2 again. The last part of our assertion uses the projective resolution  $C^{*,*}$  of  $k_\bullet^\vee$  developed in [1, Section 4] which actually is a double complex. Note that since  $(\cdot)^\vee$  provides an isomorphism of categories,  $(C^{*,*})^\vee$  is a projective resolution of  $k_\bullet$ . We observe that for this specific resolution, the homology of the double complex  $\mathrm{Hom}_{k[\Lambda]}((C^{*,*})^\vee, Y_\bullet)$  is exactly the cocyclic double complex of the cocyclic  $k$ -module  $Y_\bullet$ .  $\square$

**Proposition 3.5.** *Let  $X_\bullet$  be a cyclic  $H$ -module. Then there are three spectral sequences with*

$$E_{p,q}^2 = HC^p(\mathrm{Tor}_q^H(X_\bullet, k)) \quad {}'E_{p,q}^2 = \mathrm{Hom}_k(\mathrm{Tor}_p^H(HC^q(X_\bullet), k), k) \quad {}''E_{p,q}^2 = \mathrm{Ext}_H^p(HC_q(X_\bullet), k)$$

converging to  $\mathrm{Ext}_{H[\Lambda]}^*(X_\bullet, k_\bullet^\vee)$ .

*Proof.* Note that one can write the right exact contravariant functor  $\mathrm{Hom}_{H[\Lambda]}(\cdot, k_\bullet^\vee)$  as a composition of two right exact functors in three different ways as

$$\mathrm{Hom}_{H[\Lambda]}(\cdot, k_\bullet^\vee) \cong \mathrm{Hom}_k\left((\cdot) \otimes_H k \otimes_{k[\Lambda]} k_\bullet, k\right) \cong \mathrm{Hom}_k\left((\cdot) \otimes_{H[\Lambda]} H_\bullet \otimes_H k, k\right) \cong \mathrm{Hom}_H\left((\cdot) \otimes_{H[\Lambda]} H_\bullet, k\right)$$

The result follows after using the fact that  $k$  is a field, Proposition 3.1, and a Grothendieck spectral sequence argument.  $\square$

**Proposition 3.6.** *Let  $Y_\bullet$  be a cocyclic  $H$ -module. Then there are two spectral sequences with*

$$E_{p,q}^2 = HC^p(\mathrm{Ext}_H^q(k, Y_\bullet)) \quad {}'E_{p,q}^2 = \mathrm{Ext}_H^p(k, HC^q(Y_\bullet))$$

converging to  $\mathrm{Ext}_{H[\Lambda]}^*(k_\bullet, Y_\bullet)$ .

*Proof.* Note that we can write the left exact functor  $\mathrm{Hom}_{H[\Lambda]}(k_\bullet, \cdot)$  as a composition of left exact functors in two different ways as

$$\mathrm{Hom}_{H[\Lambda]}(k_\bullet, \cdot) \cong \mathrm{Hom}_{k[\Lambda]}(k_\bullet, \mathrm{Hom}_H(k, \cdot)) \cong \mathrm{Hom}_H(k, \mathrm{Hom}_{H[\Lambda]}(H_\bullet, \cdot))$$

The result follows after using Proposition 3.4 another Grothendieck spectral sequence argument, this time for composition of left exact functors.  $\square$

**Proposition 3.7.** *There are two spectral sequences with*

$$E_2^{p,q} = HC^p(\mathrm{Tor}_q^H(Y_\bullet^\vee, k)) \quad {}'E_2^{p,q} = \mathrm{Ext}_H^p(HC_q(Y_\bullet^\vee), k)$$

converging to  $\mathrm{Ext}_{H[\Lambda]}^*(Y_\bullet, k_\bullet)$ .

*Proof.* The proof of the first claim follows after observing the following sequence of isomorphism of functors

$$\mathrm{Hom}_{H[\Lambda]}(\cdot, k_\bullet) \cong \mathrm{Hom}_{H[\Lambda]}((\cdot)^\vee, k_\bullet^\vee) \cong \mathrm{Hom}_k \left( (\cdot)^\vee \otimes_{H[\Lambda]} k_\bullet, k \right) \cong \mathrm{Hom}_k \left( (\cdot)^\vee \otimes_H k \otimes_{k[\Lambda]} k_\bullet, k \right)$$

by using a Grothendieck spectral sequence argument. For the second claim we observe another sequence of isomorphisms

$$\mathrm{Hom}_{H[\Lambda]}(\cdot, k_\bullet) \cong \mathrm{Hom}_k \left( (\cdot)^\vee \otimes_{H[\Lambda]} k_\bullet, k \right) \cong \mathrm{Hom}_H \left( (\cdot)^\vee \otimes_{H[\Lambda]} H_\bullet, k \right)$$

and we use another Grothendieck spectral sequence.  $\square$

Let  $\mathcal{A}$  be one of  $\mathcal{P}(H)$ ,  $H[\Lambda_\mathbb{Z}]$ ,  $H[\Lambda]$ ,  $H[\mathcal{D}_\mathbb{N}]$  or  $H[\mathcal{D}_C]$ .

**Proposition 3.8.** *There is a morphism of algebras of the form  $\mathcal{A} \xrightarrow{\lambda} H \otimes \mathcal{A}$ . This yields two functors  $L_\lambda : H \otimes \mathcal{A}\text{-Mod} \rightarrow \mathcal{A}\text{-Mod}$  and  $R_\lambda : \text{Mod-}H \otimes \mathcal{A} \rightarrow \text{Mod-}\mathcal{A}$ .*

*Proof.* We will give the proof for  $\mathcal{P}(H)$ . The proofs for the other cases are similar. We define  $\lambda$  on the generators by

$$\lambda(\partial_j^n) = 1_H \otimes \partial_j^n \quad \lambda(\sigma_i^n) = 1_H \otimes \sigma_i^n \quad \lambda(\tau_n^\ell) = 1_H \otimes \tau_n^\ell \quad \lambda(h) = h_{(1)} \otimes h_{(2)}$$

for any  $n, m \geq 0$ ,  $\ell \in \mathbb{Z}$  and  $h \in H$ . Since  $H$  is a bialgebra, we know that  $H \xrightarrow{\Delta} H \otimes H$  is a morphism of  $k$ -algebras. This proves the first assertion. As for the second assertion, given a left  $H \otimes \mathcal{A}$ -module  $X_\bullet$  we define  $L_\lambda(X_\bullet) = X_\bullet$  on the object level. We define a left  $\mathcal{A}$ -module structure on  $L_\lambda(X_\bullet)$  by letting

$$\Psi(x) := (\Psi_{(-1)} \otimes \Psi_{(0)})(x)$$

for any  $x \in X_\bullet$  and for any  $\Psi \in \mathcal{A}$  where we use Sweedler's notation  $\lambda(\Psi) = (\Psi_{(-1)} \otimes \Psi_{(0)})$ .  $\square$

**Theorem 3.9.** *For any left (or right)  $\mathcal{A}$ -modules  $X_\bullet$  and  $Y_\bullet$  the bivariate cohomology  $\mathrm{Ext}_{\mathcal{A}}^*(X_\bullet, Y_\bullet)$  is a graded  $\mathrm{Ext}_H^*(k, k)$ -module.*

*Proof.* We have an action

$$\mathrm{Ext}_H^p(k, k) \otimes \mathrm{Ext}_{\mathcal{A}}^q(X_\bullet, Y_\bullet) \rightarrow \mathrm{Ext}_{H \otimes \mathcal{A}}^{p+q}(k \otimes X_\bullet, k \otimes Y_\bullet) \xrightarrow{L_\lambda^*} \mathrm{Ext}_{\mathcal{A}}^{p+q}(X_\bullet, Y_\bullet)$$

The action is associative since  $\mathcal{A} \xrightarrow{\lambda} H \otimes \mathcal{A}$  is a coassociative coaction and  $L_\lambda(k_\bullet \otimes X_\bullet) \cong X_\bullet$  for any  $\mathcal{A}$ -module  $X_\bullet$ .  $\square$

**Proposition 3.10.**  *$\mathcal{A}$  is a non-unital, non-counital bialgebra.*



*Proof.* We will give the proof for  $\mathcal{A} = \mathcal{P}(H)$ . The proofs for the other cases are similar. We define a comultiplication  $\Delta$  by defining it only on the generators. We let  $\Delta(h) = h_{(1)} \otimes h_{(2)}$  for any  $h \in H$ . The rest of the generators are group-like, i.e.

$$\Delta(\partial_i^n) = \partial_i^n \otimes \partial_i^n \quad \Delta(\sigma_j^n) = \sigma_j^n \otimes \sigma_j^n \quad \Delta(\tau_n^\ell) = \tau_n^\ell \otimes \tau_n^\ell$$

for all possible  $m, n, i, j, \ell$ . It is routine to check that the comultiplication gives us a morphism of algebras  $\mathcal{A} \xrightarrow{\Delta} \mathcal{A} \otimes \mathcal{A}$ .  $\square$

**Theorem 3.11.**  $\text{Ext}_{\mathcal{A}}^*(k_\bullet, k_\bullet)$  is a graded  $k$ -algebra and the bivariant cohomology groups  $\text{Ext}_{\mathcal{A}}^*(X_\bullet, Y_\bullet)$  form a graded  $k$ -module over  $\text{Ext}_{\mathcal{A}}^*(k_\bullet, k_\bullet)$ .

*Proof.* The proof is very similar to the proof of Theorem 3.9 after observing that  $\mathcal{A}$  is a bialgebra is equivalent to the fact that  $\mathcal{A} \xrightarrow{\Delta} \mathcal{A} \otimes \mathcal{A}$  is a coassociative coaction and that  $L_\Delta(k_\bullet \otimes X_\bullet)$  is naturally isomorphic to  $X_\bullet$  as  $\mathcal{A}$ -modules.  $\square$

**Corollary 3.12.** *There is a spectral sequence of graded algebras with*

$$E_2^{p,q} = \text{Ext}_H^p(k, k) \otimes HC^q(k_\bullet)$$

*converging to  $\text{Ext}_{H[\Lambda]}^*(k_\bullet, k_\bullet)$ .*

*Proof.* We will use the spectral sequence in Proposition 3.6. Note that  $\text{Ext}_{H[\Lambda]}^*(H_\bullet, k_\bullet)$  is the ordinary cyclic cohomology of  $k$  viewed as a coalgebra. Therefore it is the polynomial algebra  $k[u]$  generated by an element of degree 2 with the trivial  $H$ -action.  $\square$

The bivariant theory we defined above is essentially an  $H$ -linear version of Connes' bivariant cyclic theory [1, 2] as studied by [15]. One can also define a Jones-Kassel [10] variant of the cohomology we developed above as follows. Instead of using the derived category of  $H[\Lambda]$ -modules, one can use the derived category of unbounded  $H$ -linear chain complexes. For the positive cyclic theory we first chose an arbitrary projective resolution  $\mathcal{R}_{\bullet,*}$  of  $k_\bullet$  (cocyclic  $H$ -module of  $k$  considered as a  $H$ -module coalgebra) in the category of cocyclic  $H$ -modules and consider the differential graded  $k$ -module  $X_\bullet \otimes_{H[\Lambda]} \mathcal{R}_{\bullet,*}$  for a cyclic  $H$ -module  $X_\bullet$  and  $\text{Hom}_{H[\Lambda]}(\mathcal{R}_{\bullet,*}, Y_\bullet)$  for a cocyclic  $H$ -module  $Y_\bullet$ . These associations are functorial. Denote these functors by  $CP_*^{JK}(X_\bullet)$  and  $CP_{JK}^*(Y_\bullet)$  respectively. As long as our (co)cyclic objects are  $H$ -(co)unital, one can replace  $C_*^{JK}(X_\bullet)$  and  $C_{JK}^*(Y_\bullet)$  by the corresponding  $(b, B)$ -complexes of  $X_\bullet$  and  $Y_\bullet$ . These complexes have an intrinsic degree  $\pm 2$  endomorphism due to the fact that cohomology of cyclic groups have such an endomorphism. This is Connes' periodicity operator  $S$ . Therefore,  $C_*^{JK}(X_\bullet)$  and  $C_{JK}^*(Y_\bullet)$  are differential graded  $H[S]$ -modules where  $\deg(S) = \pm 2$  depending on whether we have a cyclic or cocyclic  $H$ -module. Then for two right  $H[\Lambda]$ -modules  $X_\bullet$  and  $X'_\bullet$  and two left  $H[\Lambda]$ -modules  $Y_\bullet$  and  $Y'_\bullet$  we define two bivariant

functors

$$\begin{aligned} HC_{JK}^*(X_\bullet, X'_\bullet) &:= \mathbf{Ext}_{H[S]}^*(CP_*^{JK}(X_\bullet), CP_*^{JK}(X'_\bullet)) \\ HC_{JK}^*(Y_\bullet, Y'_\bullet) &:= \mathbf{Ext}_{H[S]}^*(CP_{JK}^*(Y_\bullet), CP_{JK}^*(Y'_\bullet)) \end{aligned}$$

where  $\mathbf{Ext}$  functors are the morphisms in the derived category of unbounded chain complexes of  $H[S]$ -modules. Since the associations  $X_\bullet \mapsto CP_*^{JK}(X_\bullet)$  and  $Y_\bullet \mapsto CP_{JK}^*(Y_\bullet)$  are functorial, we have well-defined morphisms, called comparison morphisms, as

$$\begin{aligned} \mathbf{Ext}_{H[\Lambda]}^*(X_\bullet, X'_\bullet) &\xrightarrow{CP_{JK}} \mathbf{Ext}_{H[S]}^*(CP_*^{JK}(X_\bullet), CP_*^{JK}(X'_\bullet)) \\ \mathbf{Ext}_{H[\Lambda]}^*(Y_\bullet, Y'_\bullet) &\xrightarrow{CP^{JK}} \mathbf{Ext}_{H[S]}^*(CP_{JK}^*(Y_\bullet), CP_{JK}^*(Y'_\bullet)) \end{aligned}$$

There are also negative and periodic versions of this bivariant cohomology theory which admit comparison morphisms, similar to the comparison morphisms we gave above, from our bivariant theory. For the negative and periodic cyclic theories we replace  $\mathcal{R}_{\bullet,*}$  with two specific unbounded differential graded cocyclic  $H[S]$ -modules.

#### 4. BIVARIANT HOPF AND BIVARIANT EQUIVARIANT CYCLIC COHOMOLOGY OF MODULE ALGEBRAS

Recall that an algebra  $A$  is called a right  $H$ -module algebra if (i)  $A$  is a right  $H$ -module and (ii) the multiplication map  $A \otimes A \rightarrow A$  is an  $H$ -module morphism, i.e.  $(a_1 a_2)h = (a_1 h_{(1)})(a_2 h_{(2)})$  for any  $h \in H$  and  $a_1, a_2 \in A$ . If  $A$  is unital, we also require  $(1_A)h = \varepsilon(h)1_A$  for any  $h \in H$ .

**Definition 4.1.** Let  $A$  be an  $H$ -module algebra and  $M$  be a right-right  $H$ -module/comodule. Define a right  $\mathcal{P}(H)$ -module  $\mathbb{T}_\bullet(A, M)$  by letting  $\mathbb{T}_n(A, M) := A^{\otimes n+1} \otimes M$  and

$$\begin{aligned} (a_0 \otimes \cdots \otimes a_n \otimes m)\tau_n &= a_n m_{(1)} \otimes a_0 \otimes \cdots \otimes a_{n-1} \otimes m_{(0)} \\ (a_0 \otimes \cdots \otimes a_n \otimes m)\partial_j^{n-1} &= \begin{cases} a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes m & \text{if } 0 \leq j \leq n-1 \\ (a_n m_{(1)})a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes m_{(0)} & \text{if } j = n \end{cases} \\ (a_0 \otimes \cdots \otimes a_n \otimes m)\sigma_i^n &= a_0 \otimes \cdots \otimes a_i \otimes 1_A \otimes a_{i+1} \otimes \cdots \otimes m \\ (a_0 \otimes \cdots \otimes a_n \otimes m)h &= (a_0 h_{(2)}) \otimes \cdots \otimes (a_n h_{(n+2)}) \otimes m h_{(1)} \end{aligned}$$

for  $h \in H$ ,  $0 \leq j \leq n$  and  $0 \leq i \leq n$ , and  $a_0 \otimes \cdots \otimes a_n \otimes m$  in  $\mathbb{T}_n(A, M)$ .

**Definition 4.2.** Let  $A$  and  $M$  be as before. We define  $Q_\bullet(A, M)$  as the largest quotient of  $\mathbb{T}_\bullet(A, M)$  which is a cyclic  $H$ -module. Equivalently, we have

$$Q_\bullet(A, M) := \mathbb{T}_\bullet(A, M) \otimes_{\mathcal{P}(H)} H[\Lambda]$$

**Definition 4.3.** Let  $A$  and  $A'$  be  $H$ -module algebras and  $M$  and  $M'$  be  $H$ -module/comodules. We define the bivariant equivariant cyclic cohomology of the pair  $(A, A')$  with coefficients in the pair  $(M, M')$  as

$$HC_H^*(A, A'; M, M') := \text{Ext}_{H[\Lambda]}^*(Q_\bullet(A, M), Q_\bullet(A', M'))$$

In the special case where  $M = M' = k$  and  $H = k$ , our definition reduces to Connes' bivariant cyclic theory [2, 15]:

$$HC_k^*(A, A'; k, k') \cong HC^*(A, A')$$

It is also immediate from our definition that we have a graded associative product, defined by composition

$$HC_H^p(A, A'; M, M') \otimes HC_H^q(A', A''; M', M'') \rightarrow HC_H^{p+q}(A, A''; M, M'')$$

In particular each bivariant group  $HC_H^*(A, A'; M, M')$  is a graded associative algebra.

In Theorem 3.9 we show that bivariant cyclic cohomology of cyclic  $H$ -modules are graded modules over the graded algebra  $\text{Ext}_H^*(k, k)$ . Specializing to our present case, we obtain

**Proposition 4.4.** *The bivariant equivariant cyclic cohomology groups  $HC_H^*(A, A'; M, M')$  are graded modules over the graded  $k$ -algebra  $\text{Ext}_H^*(k, k)$ .*

**Corollary 4.5.** *Let  $G$  be a group and let  $A$  and  $A'$  be  $G$ -algebras. Then the bivariant equivariant cyclic cohomology groups  $HC_{k[G]}^*(A, A'; M, M')$  are graded modules over the group cohomology algebra  $H^*(G, k)$  for any pair of  $G$ -module/comodules  $M, M'$ .*

**Corollary 4.6.** *Let  $\mathfrak{g}$  be a Lie algebra and let  $A$ , and  $A'$  be  $U(\mathfrak{g})$ -module algebras, i.e.  $\mathfrak{g}$  acts on  $A$  and  $A'$  by derivations. Then the bivariant equivariant cyclic cohomology groups  $HC_{U(\mathfrak{g})}^*(A, A'; M, M')$  are graded modules over the Lie algebra cohomology  $H^*(\mathfrak{g}, k)$  for any pair of  $U(\mathfrak{g})$ -module/comodules  $M, M'$ .*

Now we shift our attention to a different kind of bivariant cyclic (co)homology of  $H$ -module algebras.

**Definition 4.7.** Given an  $H$ -module algebra  $A$  and an  $H$ -module/comodule  $M$  we define a cyclic  $k$ -module

$$\mathcal{C}_\bullet(A, M) := Q_\bullet(A, M) \otimes_H k$$

With this definition at hand, we define Hopf cyclic homology and cohomology of the triple  $(A, H, M)$  by

$$HC_*^{\text{Hopf}}(A, M) := \text{Tor}_*^{k[\Lambda]}(\mathcal{C}_\bullet(A, M), k_\bullet) \quad HC_{\text{Hopf}}^*(A, M) := \text{Ext}_{k[\Lambda]}^*(\mathcal{C}_\bullet(A, M), k_\bullet^\vee)$$

Note that, if  $H = k$  and  $M = k$ , this definition reduces to the ordinary cyclic (co)homology of algebras. Note also that we can define another bivariant cyclic theory, called bivariant Hopf cyclic cohomology, of  $H$ -module algebras as

$$HC_{\text{Hopf}}^*(A, A'; M, M') := \text{Ext}_{k[\Lambda]}^*(\mathcal{C}_\bullet(A, M), \mathcal{C}_\bullet(A', M'))$$

for any pair of  $H$ -module algebras  $A$  and  $A'$  and any pair of stable  $H$ -module/comodules  $M$  and  $M'$ .

## 5. BIVARIANT HOPF AND BIVARIANT EQUIVARIANT CYCLIC COHOMOLOGY OF MODULE COALGEBRAS

Recall that a coalgebra  $C$  is called a left  $H$ -module coalgebra if (i)  $C$  is a left  $H$ -module and (ii) the comultiplication map  $C \xrightarrow{\Delta} C \otimes C$  is an  $H$ -module morphism, i.e. one has  $\Delta(h(c)) = h_{(1)}(c_{(1)}) \otimes h_{(2)}(c_{(2)})$  for any  $h \in H$  and  $c \in C$ . Also, if  $C$  is counital then we require  $\varepsilon(h(c)) = \varepsilon(h)\varepsilon(c)$  for any  $h \in H$  and  $c \in C$ .

**Definition 5.1.** Let  $C$  be a left  $H$ -module coalgebra and  $M$  be a left-left  $H$ -module/comodule. We define a left  $\mathcal{P}(H)$ -module  $\mathbb{T}_\bullet(C, M)$  by letting

$$\mathbb{T}_n(C, M) = C^{\otimes n+1} \otimes M$$

with the  $\mathcal{P}(H)$ -action defined as

$$\begin{aligned} \tau_n(c^0 \otimes \cdots \otimes c^n \otimes m) &= c^1 \otimes \cdots \otimes c^n \otimes m_{(-1)} c^0 \otimes m_{(0)} \\ \partial_j^n(c^0 \otimes \cdots \otimes c^n \otimes m) &= \begin{cases} c^0 \otimes \cdots \otimes c_{(1)}^j \otimes c_{(2)}^j \otimes \cdots \otimes m & \text{if } 0 \leq j \leq n \\ c_{(2)}^0 \otimes c^1 \otimes \cdots \otimes c^n \otimes m_{(-1)} c_{(1)}^0 \otimes m_{(0)} & \text{if } j = n+1 \end{cases} \\ \sigma_i^{n-1}(c^0 \otimes \cdots \otimes c^n \otimes m) &= \varepsilon(c^i)(c^0 \otimes \cdots \otimes c^{i-1} \otimes c^{i+1} \otimes \cdots \otimes c^n \otimes m) \\ h(c^0 \otimes \cdots \otimes c^n \otimes m) &= h_{(1)}(c_0) \otimes \cdots \otimes h_{(n+1)}(c_n) \otimes h_{(n+2)}m \end{aligned}$$

defined for  $0 \leq j \leq n+1$ ,  $0 \leq i \leq n-1$ ,  $h \in H$  and  $c^0 \otimes \cdots \otimes c^n \otimes m$  from  $\mathbb{T}_n(C, M)$ .

**Definition 5.2.** Assume  $C$ ,  $H$ , and  $M$  are as above. We define two cocyclic  $k$ -modules

$$Q_\bullet(C, M) := H[\Lambda] \underset{\mathcal{P}(H)}{\otimes} \mathbb{T}_\bullet(C, M) \quad \mathcal{C}_\bullet(C, M) := k[\Lambda] \underset{\mathcal{P}(H)}{\otimes} \mathbb{T}_\bullet(C, M)$$

The latter can also be defined as  $\mathcal{C}_\bullet(C, M) = k \underset{H}{\otimes} Q_\bullet(C, M)$ .

**Definition 5.3.** We define the bivariant Hopf cyclic cohomology groups of a pair of  $H$ -module coalgebras  $C, C'$  and a pair of  $H$ -module/comodules  $M, M'$  as

$$HC_{\text{Hopf}}^*(C, C'; M, M') = \text{Ext}_{k[\Lambda]}^*(\mathcal{C}_\bullet(C, M), \mathcal{C}_\bullet(C', M'))$$

Recall from [9] that an arbitrary right-right (resp. left-left)  $H$ -module/comodule  $M$  is called stable if one has  $m_{(0)}m_{(1)} = m$  (resp.  $m_{(-1)}m_{(0)} = m$ ) for any  $m \in M$ . When  $H$  is a Hopf algebra with an invertible antipode,  $M$  is called an anti-Yetter-Drinfeld module if

$$(hm)_{(-1)} \otimes (hm)_{(0)} = h_{(1)}m_{(-1)}S^{-1}(h_{(3)}) \otimes h_{(2)}m_{(0)}$$

for any  $h \in H$  and  $m \in M$ . Following [9], we will refer stable anti-Yetter-Drinfeld modules as SAYD-modules.

When  $H$  is a bialgebra, in [11] for an  $H$ -module coalgebra  $C$  and a stable  $H$ -module/comodule  $M$ , we defined a para-cocyclic  $H$ -module  $\mathbb{PCM}_*(C, H, M)$  as the largest quotient of  $\mathbb{T}_\bullet(C, M)$  which is a para-cocyclic  $H$ -module. In other words  $\mathbb{PCM}_*(C, H, M) := H[\Lambda_{\mathbb{Z}}] \otimes_{\mathcal{P}(H)} \mathbb{T}_\bullet(C, M)$ . Then we used the product  $k \otimes_H \mathbb{PCM}_*(C, H, M)$  to compute the bialgebra cyclic homology of the triple  $(C, H, M)$ . It is also shown in [11] that when  $H$  is a Hopf algebra with an invertible antipode and  $M$  is an SAYD module this complex reduces to the Hopf cyclic complex triple  $(C, H, M)$  [8].

**Lemma 5.4.** *Let  $C$  be an  $H$ -module coalgebra and  $M$  be a stable  $H$ -module/comodule. Then one has an isomorphism of cocyclic  $k$ -modules  $k \otimes_H \mathbb{PCM}_*(C, H, M) \cong \mathcal{C}_\bullet(C, M)$ . In other words,  $\mathcal{C}_\bullet(C, M)$  is isomorphic to the cocyclic  $k$ -module which yields bialgebra cyclic homology in [11].*

*Proof.* Notice that  $\tau_n^{n+1}(c^0 \otimes \cdots \otimes c^n \otimes m) = m_{(-1)}(c^0 \otimes \cdots \otimes c^n \otimes m_{(0)})$  for any  $n \geq 0$  and  $(c^0 \otimes \cdots \otimes c^n \otimes m)$  in  $Q_\bullet(C, M)$ .  $\square$

Now, Lemma 5.4 implies that if  $C = H$  is a Hopf algebra and  $M$  is a SAYD module then  $\mathcal{C}_\bullet(H, M)$  is isomorphic to the cocyclic  $k$ -module of [8] which is used to compute Hopf cyclic cohomology. Which in turn means if  $H$  is a Hopf algebra which admits a modular pair  $(\sigma, \delta)$  in involution [4] and  $M = k_{(\sigma, \delta)}$  is the 1-dimensional  $H$ -module/comodule associated with the modular pair  $(\sigma, \delta)$  in involution, then  $\mathcal{C}_\bullet(H, k_{(\sigma, \delta)})$  is the cocyclic module of [4, 6] which computes Hopf cyclic cohomology of such a Hopf algebra.

**Theorem 5.5 (Specialization theorem).** *Let  $C$  be an  $H$ -module coalgebra and  $M$  be a stable  $H$ -module/comodule. Bivariant Hopf cyclic cohomology groups  $HC_{\text{Hopf}}^*(k, C; k, M)$  are the Hopf cyclic cohomology groups of the triple  $(C, H, M)$ . Bivariant Hopf cyclic cohomology groups  $HC_{\text{Hopf}}^*(C, k; M, k)$  are the cyclic cohomology groups of the dual cyclic module  $\mathcal{C}_\bullet(C, M)^\vee$ .*

*Proof.* Proof easily follows from Proposition 3.4 and Lemma 5.4.  $\square$

As in the case of ordinary bivariant cyclic cohomology, the bivariant Hopf cyclic cohomology carry an additional structure. The proof of the following Proposition is the same as its ordinary counterpart in cyclic homology of algebras [15, 1].

**Proposition 5.6.** *Bivariant Hopf cyclic cohomology groups  $HC_{\text{Hopf}}^*(C, C'; M, M')$  of a pair of  $H$ -module coalgebras  $C, C'$  and a pair of stable  $H$ -module/comodules  $M, M'$  are graded modules over the polynomial algebra  $k[u] = \text{Ext}_{k[\Lambda]}(k_\bullet, k_\bullet)$  where  $\deg(u) = 2$ .*

Now we shift our attention to a different kind of a bivariant cyclic (co)homology of  $H$ -module coalgebras.

**Definition 5.7.** For a pair of  $H$ -module coalgebras  $C$  and  $C'$  and a pair of  $H$ -module/comodules  $M$  and  $M'$  we define bivariant equivariant cyclic homology

$$HC_H^*(C, C'; M, M') := \text{Ext}_{H[\Lambda]}^*(Q_\bullet(C, M), Q_\bullet(C', M'))$$

**Proposition 5.8.** *Let  $C$  be an  $H$ -module coalgebra and  $M$  be a  $H$ -module/comodule. Then there is a spectral sequence with*

$$E_2^{p,q} = HC^p(\text{Ext}_H^q(k, Q_\bullet(C, M)))$$

*converging to the bivariant equivariant cyclic cohomology groups  $HC_H^*(k, C; k, M)$ . In particular, if  $H$  is semi-simple then  $HC_H^*(k, C; k, M)$  is the Hopf cyclic homology of the pair  $(C, M)$ .*

*Proof.* Proposition 3.6. □

**Proposition 5.9.** *Let  $C$  be an  $H$ -module coalgebra and  $M$  be a  $H$ -module/comodule. Then there is a spectral sequence with*

$$E_{p,q}^2 = HC^p(\text{Tor}_q^H(Q_\bullet(C, M)^\vee, k))$$

*converging to the bivariant equivariant cyclic cohomology groups  $HC_H^*(C, k; M, k)$ . In particular for  $q = 0$  we see that  $E_{p,0}^2$  is the dual Hopf cyclic cohomology of the triple  $(C, H, M)$ .*

*Proof.* We use Proposition 3.7 to get the spectral sequence. Second assertion follows from Lemma 5.4. □

The following Proposition is an immediate consequence of the Theorem 3.9.

**Proposition 5.10.** *The bivariant equivariant cohomology groups  $HC_H^*(C, C'; M, M')$  are graded modules over the graded algebra  $\text{Ext}_H^*(k, k)$ .*

**Corollary 5.11.** *Let  $G$  be a group and let  $C$  be a  $G$ -coalgebra. Then the bivariant equivariant cyclic cohomology modules  $HC_{k[G]}^*(C, C'; M, M')$  are graded modules over the group cohomology  $H^*(G, k)$  for any pair  $C, C'$  of  $k[G]$ -module coalgebras and any pair  $M, M'$  of  $k[G]$ -module/comodules.*

**Corollary 5.12.** *Let  $\mathfrak{g}$  be a Lie algebra and let  $C$  be a  $U(\mathfrak{g})$ -module coalgebra. Then the bivariant equivariant cyclic cohomology modules  $HC_{U(\mathfrak{g})}^*(C, C'; M, M')$  are graded modules over the Lie algebra cohomology  $H^*(\mathfrak{g}, k)$  for any pair  $C, C'$  of  $U(\mathfrak{g})$ -module coalgebras and any pair  $M, M'$  of  $U(\mathfrak{g})$ -module/comodules.*

## 6. CYCLIC (CO)HOMOLOGY OF $H$ -CATEGORIES AND MORITA INVARIANCE

Let  $H$  be an arbitrary Hopf algebra. Our goal in this section is to extend the formalism of Hopf cyclic cohomology to a certain class of additive categories that we call  $H$ -categories. We show that if an  $H$ -category  $\mathcal{C}$  is separated over a subcategory  $\mathcal{E}$ , the Hopf cyclic and equivariant cyclic (co)homology groups of  $\mathcal{C}$  and  $\mathcal{E}$  are isomorphic. From this one easily derives Morita invariance theorems for Hopf cyclic and equivariant cyclic (co)homology theories we developed previous sections.

**Definition 6.1.** A small  $k$ -linear category  $\mathcal{C}$  is called an  $H$ -category if (i) for every pair of objects  $X$  and  $Y$  the set of morphisms  $\text{Hom}_{\mathcal{C}}(X, Y)$  is a left  $H$ -module, (ii)  $h(id_X) = \varepsilon(h)id_X$  for any  $X \in \text{Ob}(\mathcal{C})$  and

$h \in H$ , and (iii) the composition of morphisms is  $H$ -equivariant, i.e.

$$h(fg) = h_{(1)}(f)h_{(2)}(g)$$

for every pair  $Z \xleftarrow{f} Y$  and  $Y \xleftarrow{g} X$  of composable morphisms. A functor of  $H$ -categories  $F : \mathcal{C} \rightarrow \mathcal{C}'$  is called an  $H$ -equivariant functor if the structure morphisms

$$F_{X,Y} : \text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{C}'}(F(X), F(Y))$$

are morphisms of  $H$ -modules, i.e.  $F_{X,Y}(h(f)) = h(F_{X,Y}(f))$  for any pair of objects  $X$  and  $Y$  in  $\mathcal{C}$  and for any  $h \in H$  and  $f \in \text{Hom}_{\mathcal{C}}(X, Y)$ .

**Definition 6.2.** A morphism  $X \xleftarrow{f} Y$  in an  $H$ -category  $\mathcal{C}$  is called  $H$ -invariant if  $h(f) = \varepsilon(h)f$  for any  $h \in H$ . The subcategory of  $H$ -invariant morphisms of  $\mathcal{C}$  is denoted by  ${}^H\mathcal{C}$ .

Our main source of examples of  $H$ -categories will come from  $H$ -equivariant modules over  $H$ -module algebras.

**Definition 6.3.** Let  $A$  be a unital  $H$ -module algebra. A left  $A$ -module  $P$  is called an  $H$ -equivariant  $A$ -module if (i)  $P$  is an  $H$ -module and (ii) the module action  $A \otimes P \rightarrow P$  is a morphism of  $H$ -modules i.e.

$$h(ax) = h_{(1)}(a)h_{(2)}(x)$$

for any  $a \in A$ ,  $h \in H$  and  $x \in P$ . Similarly, one can define  $H$ -equivariant left  $A$ -modules and  $H$ -equivariant  $A$ -bimodules.

**Example 6.4.** Let  $H$  be a Hopf algebra and  $A$  be an arbitrary  $H$ -module algebra. Let  $\mathbf{mod}_H\text{-}A$  be the category of all finitely generated  $H$ -equivariant right  $A$ -modules with all morphisms of right  $A$ -modules between them. For a pair of objects  $X$  and  $Y$  we define an  $H$ -action on  $\text{Hom}_A(X, Y)$  by letting  $h(f)(x) = h_{(1)}f(S(h_{(2)}))x$  for any  $x \in X$  and  $f \in \text{Hom}_k(X, Y)$ . It is easy to check that  $h(f)$  is still a morphism of right  $A$ -modules:

$$\begin{aligned} (hf)(xa) &= h_{(1)}f(S(h_{(2)}))(xa) = h_{(1)}f(S(h_{(3)})(x)S(h_{(2)})(a)) \\ &= h_{(1)}f(S(h_{(4)})(x))h_{(2)}S(h_{(3)})(a) = (hf)(x)a \end{aligned}$$

for any  $x \in X$ ,  $a \in A$  and  $h \in H$ . So  $\mathbf{mod}_H\text{-}A$  is an  $H$ -category. Note that  ${}^H\mathbf{mod}_H\text{-}A$  is the subcategory of finitely generated  $H$ -equivariant  $A$ -modules with  $H$ -linear and  $A$ -linear morphisms.

**Example 6.5.** Let  $H$  and  $A$  be as before. One can also consider  $\mathbf{free}_H\text{-}A$  the category of finitely generated free  $A$ -modules, which are  $H$ -equivariant  $A$ -modules automatically. There is also  $*_H^A$  the category which consists of only one object  $A$  considered as an  $H$ -equivariant right  $A$ -module.

**Example 6.6.** A projective  $A$ -module  $P$  is called  $H$ -equivariantly projective if (i) there exists another  $A$ -module  $Q$  such that  $F = P \oplus Q$  is a free  $A$ -module (with the inherent  $H$ -equivariant  $A$ -module structure) and (ii) the canonical epimorphism  $F \xrightarrow{\pi_P} P$  is  $H$ -invariant. The subcategory of  $\mathbf{mod}_H A$  which consists of finitely generated  $H$ -equivariantly projective right  $A$ -modules is denoted by  $\mathbf{proj}_H A$ .

**Definition 6.7.** Let  $\mathcal{C}$  be an  $H$ -category and  $M$  be a stable  $H$ -module/comodule. We define a pseudo-para-cyclic module (i.e. a right  $\mathcal{P}(H)$ -module)  $\mathbb{T}_\bullet(\mathcal{C}, M)$  by letting

$$\mathbb{T}_n(\mathcal{C}, M) = \bigoplus_{X_0, \dots, X_n} \mathrm{Hom}_{\mathcal{C}}(X_1, X_0) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{C}}(X_n, X_{n-1}) \otimes \mathrm{Hom}_{\mathcal{C}}(X_0, X_n) \otimes M$$

We define structure morphisms as follows:

$$\begin{aligned} (f_0 \otimes \cdots \otimes f_n \otimes m) \tau_n &= m_{(-1)} f_n \otimes f_0 \otimes \cdots \otimes f_{n-1} \otimes m_{(0)} \\ (f_0 \otimes \cdots \otimes f_n \otimes m) \partial_j^{n-1} &= \begin{cases} f_0 \otimes \cdots \otimes f_j f_{j+1} \otimes \cdots \otimes f_n \otimes m & \text{if } 0 \leq j \leq n-1 \\ (m_{(-1)} f_n) f_0 \otimes f_1 \otimes \cdots \otimes f_{n-1} \otimes m_{(0)} & \text{if } j = n \end{cases} \\ (f_0 \otimes \cdots \otimes f_n \otimes m) \sigma_j^n &= \begin{cases} f_0 \otimes \cdots \otimes f_j \otimes id_{X_{j+1}} \otimes f_{j+1} \otimes \cdots \otimes f_n \otimes m & \text{if } 0 \leq j \leq n-1 \\ f_0 \otimes \cdots \otimes f_n \otimes id_{X_0} \otimes m & \text{if } j = n \end{cases} \\ (f_0 \otimes \cdots \otimes f_n \otimes m) h &= S^{-1}(h_{(n+2)})(f_0) \otimes \cdots \otimes S^{-1}(h_{(2)})(f_n) \otimes S^{-1}(h_{(1)})(m) \end{aligned}$$

for any  $n \geq 0$ ,  $0 \leq j \leq n$ ,  $h \in H$ , and  $f_0 \otimes \cdots \otimes f_n \otimes m \in \mathbb{T}_n(\mathcal{C}, M)$ .

**Definition 6.8.** For an  $H$ -category  $\mathcal{C}$  and an stable  $H$ -module/comodule  $M$  we define  $Q_\bullet(\mathcal{C}, M)$  as the largest quotient of  $\mathbb{T}_\bullet(\mathcal{C}, M)$  which is a cyclic  $H$ -module. In other words we divide  $\mathbb{T}_\bullet(\mathcal{C}, M)$  by the graded  $H$ -submodule and a cyclic  $k$ -module generated by elements of the form

$$(\Psi)[x, \tau_n^i] + (\Phi)(\tau_n^{n+1} - \tau_n^0)$$

for  $x \in H$ ,  $n, i \in \mathbb{N}$  and  $\Psi, \Phi \in \mathbb{T}_n(\mathcal{C}, M)$ . Equivalently we have

$$Q_\bullet(\mathcal{C}, M) := \mathbb{T}_\bullet(\mathcal{C}, M) \otimes_{\mathcal{P}(H)} H[\Lambda]$$

Assume  $\mathcal{A}_*$  and  $\mathcal{B}_*$  are two pre-simplicial  $H$ -modules (i.e. right  $H[\mathcal{D}]$ -modules), which can also be considered as differential graded  $H$ -modules. Assume that  $f, g : \mathcal{A}_* \rightarrow \mathcal{B}_*$  are morphisms of pre-simplicial  $H$ -modules. Recall that if there is a set of  $H$ -module morphisms  $h_j : \mathcal{A}_n \rightarrow \mathcal{B}_{n+1}$  defined for  $0 \leq j \leq n$  satisfying

$$h_j \partial_i = \partial_i h_{j+1}, \quad \text{if } i \leq j \qquad h_j \partial_i = \partial_{i+1} h_j, \quad \text{if } i \geq j+1 \qquad \partial_i h_i = \partial_i h_{i-1}, \quad \text{if } i \geq 1$$

where  $f_n = \partial_0 h_0$  and  $g_n = \partial_{n+1} h_n$  for any  $n \geq 0$ , then  $s_n = \sum_{i=0}^n (-1)^i h_i$  defines a chain homotopy between the morphism of differential graded  $H$ -modules  $f_*$  and  $g_*$ .



For any (co)cyclic  $H$ -module  $X_\bullet$  (i.e. a right and resp. left  $H[\Lambda]$ -module), we will use  $X_\bullet^{\mathcal{D}}$  to denote the same (co)cyclic  $H$ -module viewed as a pre-(co)simplicial module (i.e. right and resp. left  $H[\mathcal{D}]$ -module). In other words,  $X_\bullet^{\mathcal{D}}$  stands for the restriction  $\text{Res}_{H[\mathcal{D}]}^{H[\Lambda]}(X_\bullet)$ .

**Lemma 6.9.** *Fix a stable  $H$ -module/comodule  $M$ . If two  $H$ -categories  $\mathcal{C}$  and  $\mathcal{C}'$  are  $H$ -equivariantly equivalent then  $Q_\bullet^{\mathcal{D}}(\mathcal{C}, M)$  and  $Q_\bullet^{\mathcal{D}}(\mathcal{C}', M)$  are  $H$ -equivariantly homotopy equivalent.*

*Proof.* Assume we have a pair of  $H$ -equivariant functors  $F : \mathcal{C} \rightarrow \mathcal{C}'$  and  $G : \mathcal{C}' \rightarrow \mathcal{C}$  such that  $\text{id}_{\mathcal{C}} \simeq GF$  and  $\text{id}_{\mathcal{C}'} \simeq FG$   $H$ -equivariantly, i.e. the natural transformations are  $H$  invariant. Call these natural transformations  $h$  and  $h'$  respectively. We will show that  $Q_\bullet^{\mathcal{D}}(GF, M) \simeq \text{id}$  on  $Q_\bullet^{\mathcal{D}}(\mathcal{C}, M)$ . The proof for  $Q_\bullet^{\mathcal{D}}(FG, M) \simeq \text{id}$  on  $Q_\bullet^{\mathcal{D}}(\mathcal{C}', M)$  is similar. Define

$$h_j(f_0 \otimes \cdots \otimes f_n \otimes m) = \begin{cases} f_0 h_{X_1}^{-1} \otimes GF(f_1) \otimes \cdots \otimes GF(f_j) \otimes h_{X_{j+1}} \otimes f_{j+1} \otimes \cdots \otimes f_n \otimes m & \text{if } 0 \leq j \leq n-1 \\ f_0 h_{X_1}^{-1} \otimes GF(f_1) \otimes \cdots \otimes GF(f_n) \otimes h_{X_0} \otimes m & \text{if } j = n \end{cases}$$

for any  $(f_0 \otimes \cdots \otimes f_n \otimes m)$  in  $Q_n(\mathcal{C}, M)$ . Note that  $\partial_0 h_0$  is the identity on  $Q_\bullet^{\mathcal{D}}(\text{id}_{\mathcal{C}}, M)$  and  $\partial_{n+1} h_n = Q_\bullet^{\mathcal{D}}(GF, M)$ . We leave the verification of the pre-simplicial homotopy identities to the reader. This proves  $Q_\bullet^{\mathcal{D}}(GF, M)$  is homotopic to  $Q_\bullet^{\mathcal{D}}(\text{id}_{\mathcal{C}}, M)$ . The proof for  $FG$  is similar.  $\square$

**Definition 6.10.** Let  $\mathcal{C}$  be an  $H$ -category and let  $\mathcal{E}$  be a full  $H$ -subcategory of  $\mathcal{C}$ . The category  $\mathcal{C}$  is called separated over  $\mathcal{E}$  iff for every object  $X$  in  $\mathcal{C}$  there exists a natural number  $n \geq 1$  and a finite set of objects  $\{X_1, \dots, X_n\}$  in  $\mathcal{E}$  such that we have  $H$ -invariant morphisms  $X_i \xleftarrow{u_i} X$  and  $X \xleftarrow{v_j} X_j$  in  $\mathcal{C}$  which satisfy  $u_i v_i = \text{id}_{X_i}$  and  $\sum_{i=1}^n v_i u_i = \text{id}_X$ .

**Proposition 6.11.**  $\text{proj}_H A$  is separated over  $\text{free}_H A$  which in turn is separated over  $*_H^A$ .

**Lemma 6.12.** *Let  $\mathcal{C}$  be an  $H$ -category separated over an  $H$ -subcategory  $\mathcal{E}$ . Then the natural inclusion of functors  $\mathcal{E} \rightarrow \mathcal{C}$  induces a homotopy equivalence of pre-simplicial  $H$ -modules  $Q_\bullet^{\mathcal{D}}(\mathcal{E}, M) \rightarrow Q_\bullet^{\mathcal{D}}(\mathcal{C}, M)$  for any stable  $H$ -module/comodule  $M$ .*

*Proof.* For every object  $X$  in  $\mathcal{C}$  fix a finite set of objects  $\{X_1, \dots, X_{n(X)}\}$  and  $H$ -invariant morphisms  $X_i \xleftarrow{u_i(X)} X$  and  $X \xleftarrow{v_j(X)} X_j$  such that the choices for every object  $X$  in  $\mathcal{E}$  are  $\{X\}$  with  $u_1(X) = \text{id}_X = v_1(X)$ . Define a morphism of graded  $H$ -modules  $Q_\bullet(\mathcal{C}, M) \xrightarrow{E_\bullet} Q_\bullet(\mathcal{E}, M)$  by letting

$$\begin{aligned} E_n(f_0 \otimes \cdots \otimes f_n \otimes m) &= \sum_{i_0, \dots, i_n} u_{i_0}(X_0) f_0 v_{i_1}(X_1) \otimes u_{i_1}(X_1) f_1 v_{i_2}(X_2) \otimes \cdots \\ &\quad \otimes u_{i_{n-1}}(X_{n-1}) f_{n-1} v_{i_n}(X_n) \otimes u_{i_n}(X_n) f_n v_{i_0}(X_0) \otimes m \end{aligned}$$

for any  $f_0 \otimes \cdots \otimes f_n \otimes m$  in  $Q_\bullet(\mathcal{C}, M)$ . One can easily check that  $E_\bullet$  is a morphism of cyclic  $H$ -modules and the composition  $Q_\bullet(\mathcal{E}, M) \xrightarrow{i_\bullet} Q_\bullet(\mathcal{C}, M) \xrightarrow{E_\bullet} Q_\bullet(\mathcal{E}, M)$  is the identity. In order to prove that  $i_\bullet E_\bullet$  and  $id_\bullet$  are homotopic on  $Q_\bullet^{\mathcal{D}}(\mathcal{C}, M)$ , we must furnish a pre-simplicial homotopy. For any  $n \geq 0$  and  $(f_0 \otimes \cdots \otimes f_n \otimes m)$  in  $Q_n(\mathcal{C}, M)$  we define

$$\begin{aligned} h_j(f_0 \otimes \cdots \otimes f_n \otimes m) \\ = \sum_{i_0, i_{j+1}, \dots, i_n} u_{i_0}(X_0) f_0 \otimes f_1 \otimes \cdots \otimes f_j \otimes v_{i_{j+1}}(X_{j+1}) \otimes u_{i_{j+1}}(X_{j+1}) f_{j+1} v_{i_{j+2}}(X_{j+2}) \otimes \cdots \\ \otimes u_{i_n}(X_n) f_n v_{i_0}(X_0) \otimes m \end{aligned}$$

for  $j \geq 0$ . For  $j = n$  we let

$$h_n(f_0 \otimes \cdots \otimes f_n \otimes m) = \sum_{i_0} u_{i_0}(X_0) f_0 \otimes f_1 \otimes \cdots \otimes f_n \otimes v_{i_0}(X_0) \otimes m$$

One can easily see that  $\partial_0 h_0 = i_\bullet E_\bullet$  and  $\partial_{n+1} h_n = id_\bullet$ . We leave the verification of the simplicial homotopy identities to the reader.  $\square$

**Definition 6.13.** Two  $H$ -module algebras  $A$  and  $A'$  are said to be  $H$ -equivariantly Morita equivalent if there is an  $H$ -equivariant  $A$ - $A'$ -bimodule  $P$  and an  $H$ -equivariant  $A'$ - $A$ -bimodule  $Q$  such that  $P \otimes_{A'} Q \cong A$  as  $H$ -equivariant  $A$ - $A$ -bimodules and  $Q \otimes_A P \cong A'$  as  $H$ -equivariant  $A'$ - $A'$ -bimodules.

**Theorem 6.14 (Equivariant Morita invariance for module algebras).** *Assume  $A$  and  $A'$  are two unital  $H$ -module algebras which are  $H$ -equivariantly Morita equivalent. Then Hopf cyclic (co)homology and equivariant cyclic (co)homology of the pairs  $(A, M)$  and  $(A', M)$  are isomorphic.*

*Proof.* Lemma 6.9 implies that pre-simplicial  $H$ -modules  $Q_\bullet^{\mathcal{D}}(\mathbf{proj}_H\text{-}A, M)$  and  $Q_\bullet^{\mathcal{D}}(\mathbf{proj}_H\text{-}A', M)$  are homotopy equivalent. The inclusion  $*_H^A \rightarrow \mathbf{proj}_H\text{-}A$  induces a homotopy equivalence  $Q_\bullet^{\mathcal{D}}(*_H^A, M) \rightarrow Q_\bullet^{\mathcal{D}}(\mathbf{proj}_H\text{-}A, M)$  by Proposition 6.11 and Lemma 6.12. Observe also that  $Q_\bullet(*_H^A, M)$  is isomorphic to  $Q_\bullet(A, M)$  as  $H[\Lambda]$ -modules. The result follows from the SBI-sequence in cyclic (co)homology and the 5-Lemma.  $\square$

**Example 6.15.** Let  $C$  be a left  $H$ -module coalgebra. A right  $C$ -comodule  $X$  is called an  $H$ -equivariant right  $C$ -comodule if the coaction  $X \xrightarrow{\rho_X} X \otimes C$  is a left  $H$ -module morphism. Explicitly, for any  $x \in X$  and  $h \in H$  one has

$$(hx)_{(0)} \otimes (hx)_{(1)} = h_{(1)}x_{(0)} \otimes h_{(2)}x_{(1)}$$

A morphism of  $C$ -comodules  $X \xrightarrow{f} Y$  is said to be  $H$ -equivariant if  $f$  is also  $H$ -linear. An arbitrary (left)  $H$ -equivariant (right)  $C$ -comodule  $F$  is said to be free if  $F$  is isomorphic to a direct sum  $C^{\oplus I}$  with its canonical  $H$ -structure where  $I$  is an arbitrary (not necessarily finite) index set. If  $I$  is finite, then  $F$  is called finitely generated. An arbitrary  $H$ -equivariant  $C$ -comodule  $P$  is said to be projective if there

exists another  $H$ -equivariant  $C$ -comodule  $Q$  such that (i)  $F := P \oplus Q$  is a free  $H$ -equivariant  $C$ -comodule (ii) the canonical epimorphism  $F \xrightarrow{\pi} C$  is  $H$ -invariant. Let  $\mathbf{mod}_H\text{-}C$  denote the category of all finitely generated  $H$ -equivariant  $C$ -comodules with not necessarily  $H$ -equivariant but all  $C$ -comodule morphisms. We will use  $\mathbf{proj}_H\text{-}C$  and  $\mathbf{free}_H\text{-}C$  to denote the subcategories of finitely generated projective and finitely generated free  $H$ -equivariant  $C$ -comodules respectively. There is also  $*_H^C$  the subcategory of  $\mathbf{free}_H\text{-}C$  on one single object  $C$  considered as an  $H$ -equivariant  $C$ -comodule via its comultiplication.

The category  $\mathbf{mod}_H\text{-}C$  and all of its subcategories we mentioned above carry an  $H$ -category structure defined as follows: given  $X \xrightarrow{f} Y$  an arbitrary  $C$ -comodule morphism between two  $H$ -equivariant  $C$ -comodules and  $h \in H$  we define

$$(hf)(x) = h_{(1)}f(S(h_{(2)})x)$$

for any  $x \in X$ . We must check that  $hf$  is still a  $C$ -comodule morphism. Recall that  $f$  is a  $C$ -comodule morphism iff  $f(x)_{(0)} \otimes f(x)_{(1)} = f(x_{(0)}) \otimes x_{(1)}$  for any  $x \in X$ . We also observe that since both  $X$  and  $Y$  are both  $H$ -equivariant  $C$ -comodules we have

$$\begin{aligned} ((hf)(x))_{(0)} \otimes ((hf)(x))_{(1)} &= h_{(1)(1)}f(S(h_{(2)})x)_{(0)} \otimes h_{(1)(2)}f(S(h_{(2)})x)_{(1)} \\ &= h_{(1)}f(S(h_{(4)})x_{(0)}) \otimes h_{(2)}S(h_{(3)})x_{(1)} \\ &= (hf)(x_{(0)}) \otimes x_{(1)} \end{aligned}$$

for any  $x \in X$  and  $h \in H$  as we wanted to show.

As before, one can show that  $\mathbf{proj}_H\text{-}C$  is separated over  $\mathbf{free}_H\text{-}C$  which in turn is separated over  $*_H^C$ . Thus Hopf and equivariant cyclic (co)homology of  $Q_\bullet(\mathbf{proj}_H\text{-}C, M)$  are the same as the Hopf and equivariant cyclic cohomology of the  $H$ -module algebra  $\text{Hom}_C(C, C)$  respectively for any stable  $H$ -module/comodule  $M$ . Now let us identify the  $H$ -module algebra  $\text{Hom}_C(C, C)$ .

For a left  $H$ -module algebra  $A$ , the opposite  $H$ -module algebra  $A^{op}$  is a right  $H$ -module algebra with  $x^{op}h := (S^{-1}(h)x)^{op}$  for any  $x \in A$  and  $h \in H$ . Note that

$$(x^{op}y^{op})h = (S^{-1}(h)(yx))^{op} = (S^{-1}(h_{(2)})yS^{-1}(h_{(1)})x)^{op} = (xh_{(1)})^{op}(yh_{(2)})^{op}$$

for any  $x^{op}, y^{op} \in A^{op}$  and  $h \in H$ .

**Proposition 6.16.** *Let  $C$  be a counital  $H$ -module coalgebra. The  $k$ -linear dual  $C^*$  is an  $H$ -module algebra with the convolution product. Moreover,  $C^*$  is isomorphic to  $\text{Hom}_C(C, C)^{op}$  the opposite  $H$ -module algebra of  $C$ -comodule endomorphisms of  $C$  as  $H$ -module algebras.*

*Proof.* Take two arbitrary elements  $\delta, \mu \in C^* = \text{Hom}_k(C, k)$  and define

$$(\delta * \mu)(c) = \delta(c_{(1)})\mu(c_{(2)})$$

for any  $c \in C$ . One can easily check that  $*$  is an associative product since  $C$  is an coassociative coalgebra. Note also that the counit  $\varepsilon$  of  $C$  is the unit of this convolution algebra. The right  $H$ -structure is given by  $(\delta h)(c) = \delta(hc)$  for any  $c \in C$  and  $h \in H$ . Now we can check that

$$((\delta * \mu)h)(c) = (\delta * \mu)(hc) = \delta(h_{(1)}c_{(1)})\mu(h_{(2)}c_{(2)}) = ((\delta h_{(1)}) * (\mu h_{(2)}))(c)$$

for any  $h \in H$  and  $c \in C$ . This proves the first assertion. For the second assertion, observe that  $\text{Hom}_C(C, C)^{op}$  is a right  $H$ -module algebra since the action of  $H$  on  $*_H^C$  was on the left. We define two  $k$ -linear morphisms  $\text{Hom}_C(C, C)^{op} \xrightarrow{\Phi} C^*$  and  $C^* \xrightarrow{\Psi} \text{Hom}_C(C, C)^{op}$  by

$$\Phi(u^{op})(c) := \varepsilon(u^{op}(c)) \quad \Psi(\delta)(c) := \delta(c_{(1)})c_{(2)}$$

for any  $u^{op} \in \text{Hom}_C(C, C)^{op}$ ,  $\delta \in C^*$  and  $c \in C$ . It is easy to see that  $\Phi(u^{op})$  is in  $C^*$  for any  $u^{op} \in \text{Hom}_C(C, C)^{op}$ . On the other hand, for  $\delta \in C^*$  we check

$$(\Psi(\delta)(c))_{(0)} \otimes (\Psi(\delta)(c))_{(1)} = \delta(c_{(1)})c_{(2)} \otimes c_{(3)} = (\Psi(\delta)(c_{(1)})) \otimes c_{(2)}$$

for any  $c \in C$ , i.e.  $\Psi(\delta)$  lies in  $\text{Hom}_C(C, C)^{op}$ . Notice also that  $\Psi$  and  $\Phi$  are inverses of each other since

$$\begin{aligned} (\Psi(\Phi(u^{op}))) (c) &= \varepsilon(u^{op}(c_{(1)}))c_{(2)} = \varepsilon(u^{op}(c)_{(1)})u^{op}(c)_{(2)} = u^{op}(c) \\ (\Phi(\Psi(\delta))) (c) &= \varepsilon(\delta(c_{(1)})c_{(2)}) = \delta(c) \end{aligned}$$

for any  $u^{op} \in \text{Hom}_C(C, C)^{op}$ ,  $\delta \in C^*$  and  $c \in C$ . For the compatibility of the  $H$ -structures we see that

$$\Phi(u^{op}h)(c) = \varepsilon(u^{op}(hc)) = \varepsilon(S^{-1}(h_{(2)})u^{op}(h_{(1)}c)) = \varepsilon(u^{op}(hc)) = (\Phi(u^{op})h)(c)$$

for any  $u^{op} \in \text{Hom}_C(C, C)^{op}$ ,  $h \in H$  and  $c \in C$ . Finally we check

$$\Psi(\delta * \mu)(c) = (\delta * \mu)(c_{(1)})c_{(2)} = \delta(c_{(1)})\mu(c_{(2)})c_{(3)} = \Psi(\mu)(\delta(c_{(1)})c_{(2)}) = (\Psi(\mu) \circ \Psi(\delta))(c)$$

where  $\circ$  denotes composition of morphisms. However, recall that in the opposite algebra the multiplication is the opposite composition. Therefore

$$\Psi(\delta * \mu) = \Psi(\delta)^{op} \Psi(\mu)^{op}$$

for any  $\delta, \mu \in C^*$  as we wanted to show.  $\square$

**Theorem 6.17 (Morita invariance for module coalgebras).** *Let  $M$  be an arbitrary stable  $H$ -module/comodule. Let  $C$  and  $C'$  be two  $H$ -module coalgebras such that the categories  $\mathbf{proj}_H\text{-}C$  and  $\mathbf{proj}_H\text{-}C'$  are  $H$ -equivariantly equivalent. Then  $H$ -module algebras  $C^*$  and  $C'^*$  have isomorphic Hopf and equivariant cyclic (co)homologies.*

**Remark 6.18.** The functor  $(\cdot)^* := \text{Hom}_k(\cdot, k)$  sends an  $H$ -equivariant  $C$ -comodule to an  $H$ -equivariant  $C^*$ -module. However, the failure of  $(\cdot)^*$  being an equivalence forbids us to translate the problem of equivariant Morita equivalence of two  $H$ -module coalgebras  $C$  and  $C'$  purely in terms of equivariant Morita

equivalence of dual  $H$ -module algebras  $C^*$  and  $(C')^*$ . Although it is true that if  $C^*$  and  $(C')^*$  are  $H$ -equivariantly Morita equivalent then  $C^*$  and  $(C')^*$  has isomorphic Hopf and equivariant cyclic cohomology, requiring  $C$  and  $C'$  to be equivariantly Morita equivalent is much weaker.

## REFERENCES

- [1] A. Connes. Cohomologie cyclique et foncteurs  $\text{Ext}^n$ . *C. R. Acad. Sci. Paris Sér. I Math.*, 296(23):953–958, 1983.
- [2] A. Connes. Noncommutative differential geometry. *Inst. Hautes Études Sci. Publ. Math.*, (62):257–360, 1985.
- [3] A. Connes. *Noncommutative geometry*. Academic Press Inc., San Diego, CA, 1994 (Also available online at <ftp://ftp.alainconnes.org/book94bigpdf.pdf>).
- [4] A. Connes and H. Moscovici. Hopf algebras, cyclic cohomology and transverse index theorem. *Comm. Math. Phys.*, 198:199–246, 1998.
- [5] A. Connes and H. Moscovici. Cyclic cohomology and Hopf algebras. *Lett. Math. Phys.*, 48(1):97–108, 1999. Moshé Flato (1937–1998).
- [6] A. Connes and H. Moscovici. Cyclic cohomology and Hopf algebra symmetry. *Lett. Math. Phys.*, 52(1):1–28, 2000.
- [7] J. Cuntz. Bivariant  $K$ - and cyclic theories. *Handbook of  $K$ -theory*. Vol. 1, 2, 655–702, Springer, Berlin, 2005.
- [8] P. M. Hajac, M. Khalkhali, B. Rangipour, and Y. Sommerhäuser. Hopf-cyclic homology and cohomology with coefficients. *C. R. Math. Acad. Sci. Paris*, 338(9):667–672, 2004.
- [9] P. M. Hajac, M. Khalkhali, B. Rangipour, and Y. Sommerhäuser. Stable anti-Yetter–Drinfeld modules. *C. R. Math Acad. Sci. Paris*, 338(8):587–590, 2004.
- [10] J. D. S. Jones and C. Kassel. Bivariant cyclic theory. *K-Theory*, 3(4):339–365, 1989.
- [11] A. Kaygun. Bialgebra cyclic homology with coefficients. *K-Theory*, 34(2):151–194, 2005.
- [12] M. Khalkhali and B. Rangipour. A new cyclic module for Hopf algebras. *K-Theory*, 27(2):111–131, 2002.
- [13] M. Khalkhali and B. Rangipour. A note on cyclic duality and Hopf algebras. *Comm. Algebra*, 33:763–773, 2005.
- [14] J-L. Loday. *Cyclic Homology*. Number 301 in Die Grundlehren der mathematischen Wissenschaften. Springer Verlag, Berlin, Heidelberg, New York, 1992.
- [15] V. Nistor. A bivariant Chern-Connes character. *Ann. of Math. (2)*, 138(3):555–590, 1993.

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